#9 Counting and Probability

- In this section we will go over principles of **Counting** which may help creating concise logical definitions and correct mathematical proofs.

Probability

S = Sample space, E = experiment

$$P(E) = \frac{N(E)}{N(S)}, \quad \sum_{E \subseteq S} P(E) = 1$$

Examples:

- 3 blue balls, 5 red balls,...
- A pair of dice is being rolled up. What is the probability that the numbers showing face up have a sum of 6?
- Toss 3 coins. WTPT there is only one tails? WTPT there are at least 2 tails? WTPT there are no tails?

Probability

Famous Example

Monty Hall Problem

«Lets Make a Deal»

https://www.youtube.com/watch?v=4Lb-6rxZxx0

Possibility Trees

Example

Teams A and B are to play each other repeatedly until one wins two games in a row or a total of three games. One way in which this tournament can be played is for A to win the first game, B to win the second, and A to win the third and fourth games. Denote this by writing A–B–A–A.

a. How many ways can the tournament be played?

b. Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

Multiplication Rule

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

$\vdots$

the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

Example: How many different PINs?

Permutations

Example

Consider the following nested loop:

for $i := 1$ **to** 4

  for $j := 1$ **to** 3

    [Statements in body of inner loop.
    None contain branching statements
    that lead out of the inner loop.]

  next j

next i

How many times will the inner loop be iterated when the algorithm is implemented and run?

Permutations

- Permutations of an n element set: $n!$
- r-permutations of an n element set: $\mathrm{P(n,r)} = \frac{n!}{(n-r)!}$
- Circular permutaitons of an n element set: (n-1)!

Permutations

Examples

a. How many ways can the letters in the word *COMPUTER* be arranged in a row?

b. How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

c. If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

a. How many different ways can three of the letters of the word *BYTES* be chosen and written in a row?

b. How many different ways can this be done if the first letter must be *B*?

Permutations

Examples

Prove the following facts: $n \geq 2$

$$P(n, 2) + P(n, 1) = n^2.$$

$$P(n + 1, 2) - P(n, 2) = 2P(n, 1).$$

$$P(n + 1, 3) = n^3 - n.$$

Addition Rule

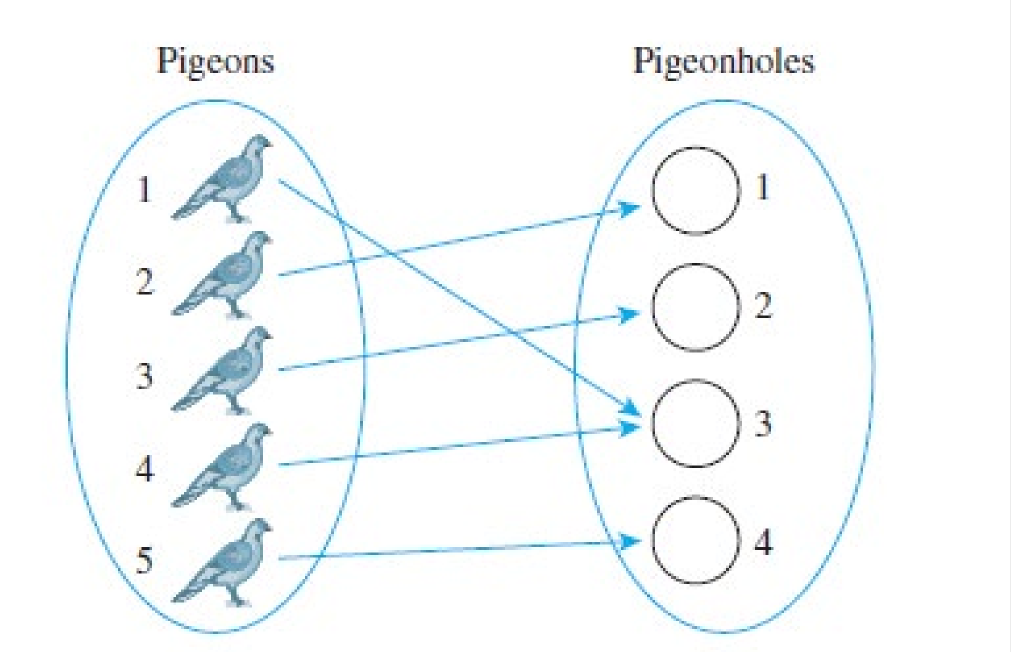
- Counting the number of elements of a union of disjoint sets
- What if the sets are not disjoint: The Inclusion/Exculison principle

Examples

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Pigeon Hole Principle



A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the range set.

a. In a group of six people, must there be at least two who were born in the same month? In a group of thirteen people, must there be at least two who were born in the same month? Why?

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

a. If five integers are selected from A, must at least one pair of the integers have a sum of 9?

b. If four integers are selected from A, must at least one pair of the integers have a sum of 9?

The shrewd guess, the fertile hypothesis, the courageous leap to a tentative conclusion—these are the most valuable coin of the thinker at work

— Jerome S. Bruner, 1960