11th Week

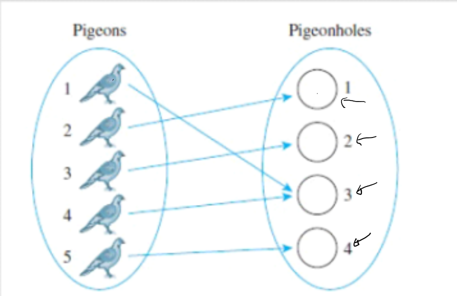
15 Aralık 2023 Cuma 09:29

Counting & Probability

Permutations ✓ → Monty Hall Problem

→ Pigeonhole Principle

Pigeon Hole Principle



$f: A \to B$ $|A| > |B|$

A function from one finite set to a smaller finite set cannot be one-to-one: There must be a least two elements in the domain that have the same image in the range set.

Generalized Pigeonhole Principle

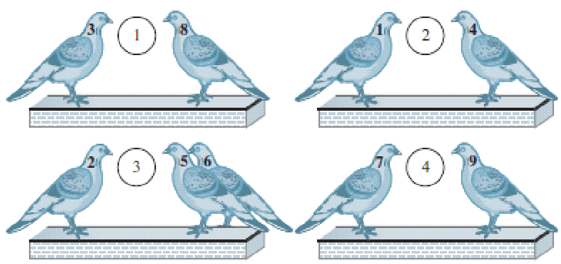
A generalization of the pigeonhole principle states that if n pigeons fly into m pigeonholes and, for some positive integer k, $k < n/m$, then at least one pigeonhole contains $k+1$ or more pigeons. This is illustrated in Figure 9.4.2 for $m = 4$, $n = 9$, and $k = 2$. Since $2 < 9/4 = 2.25$, at least one pigeonhole contains three $(2+1)$ or more pigeons. (In this example, pigeonhole 3 contains three pigeons.)

$|A| > |B|$

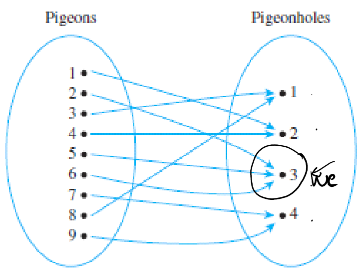
$f: A \to B$

$|A| = n$ $|B| = m$

$k+1$ $k < n/m$ pigeon



(a)



9 pigeons (b) 4 holes

Figure 9.4.2

we are trying to put least number of pigeons in each hole.

> **Generalized Pigeonhole Principle**
>
> For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k+1$ distinct elements of X.

$$9 \div 4 = 2, \text{ remainder } 1$$

$2 \to$ the least number of pigeons that can be in each hole

$k < 9/4$ $\boxed{k=2}$ $\frac{9}{4} = 2, \ldots$

9 pigeon → 4 holes $n/m = 9/4$

ex/ At least 1 hole consists of at least 3 pigeons. $k=2$, $k+1$

Ex/

→ There are 42 students who are to share 12 computers. Each student uses exactly 1 computer, and no computer is used by more than 6 students. Show that at least 5 computers are used by 3 or more students.

42 students → Lab 12 computers

Computer Labs

1 computer can be shared by at most 6 students.

$42 \div 12 = 3 \ldots$ $k < \frac{42}{12}$

by at most ...

$42 \mid 12$
$3, \ldots$

$k < \frac{42}{12}$

$k = 3$ 4

Try to prove it with contradiction

5 ...

p : { At least 5 computers are going to be used by at least 3 students }

$\rightarrow \sim p$: { At most 4 computers are being used by at least 3 students }

Assume this $\rightarrow$

1, 2, 3, 4

the boundary case $\longrightarrow$

$4 \times 6 = 24$ students

computer — each shared by 6 students

$12 - 4 = 8$ computers

$8 \times 2 = 16$ students.

$24 + 16 = 40$ students ✗

this can not reach to 42.

Combinations: Selection of elements from a set

Creating subsets

ex/ How many 2-elements subsets are there for a 4-element set?

$n = 4$ $\quad r = 2$

$$C(4,2) = \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4.3}{2} = 6$$

4 choose 2

$$C(n,r) = \binom{n}{r} = \frac{n!}{\underbrace{(n-r)}_{k}!\,r!}$$

n choose r

$\boxed{n = r + k}$

$$* \quad \binom{n}{r} = \binom{n}{k} \longrightarrow \frac{n!}{\underbrace{(n-k)}_{r}!\,k!}$$

$$\binom{7}{3} = \binom{7}{4} \qquad \binom{8}{2} = \binom{8}{6}$$

ex/ A group with 12 people will create a sub group with 5 people with respect to the cases below. In how many ways can we create this subgroup

a) Person A and Person B should be together in the group.

$12 - 2$

10

{ _, _, _, _, _ }

A B

10 people $\rightarrow$ 3

$$\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{10.\cancel{9}.\cancel{8}.\cancel{7!}}{\cancel{7!}.\cancel{3}.\cancel{2}.1} = 120$$

b) Person A and Person B should not be in this group.

$12 - 2 = 10$

$$\binom{10}{5} = \frac{10!}{5!\,5!} = \frac{\cancel{10}.9.\cancel{8}.7.\cancel{6}.\cancel{5!}}{5!\;\cancel{5!}} = 36.7 = 252$$

r- Combinations with Repetitions Allowed.

Tekrarlı r'li kombinasyon.

I will buy n pencils

$n = 15$

ex pencils: rotring (6), faber-castel (9), person, abc, def

In how many way can

ex

pencils

rotring | faber-castel | person | abc | def

equivalent rotrings here

r categories

r = 5

n = 15

In how many way can you buy these 15 pencils

Bar/Stick - cross method.

| separator between categories

X each pencil.

r-1 bars

n

n + r-1

XX | XXX XXX | XX X | X | XXX

rotring faber person abc def

choosing 4 places for my bars

r = 5 n = 15

$$\binom{n+r-1}{n} = \binom{n+r-1}{r-1}$$

$$\binom{15+4}{4} = \binom{19}{4}$$

a) At least 6 of the pencil I will buy should be Faber-Castel.

$\underbrace{XXXXXX}_{\text{reserved for faber-castel}}$ → from 5 categories buy 9

15 - 6 = 9

n = 9 r = 5 r - 1 = 4

$$\binom{9+4}{4} = \boxed{\binom{13}{4}} = \frac{13!}{\ldots}$$

b) What if there are only 5 rotrings left in the stationary shop?

≡ You may buy at most 5 rotrings.

6 →

1 rot + 2 rot. + 3 rot + 4 rot + 5 rot.

at least 6

$$\binom{19}{4} - \binom{13}{4}$$

all possibilities

at least 6 case for any category