MAT 203E DISCRETE MATHEMATICS

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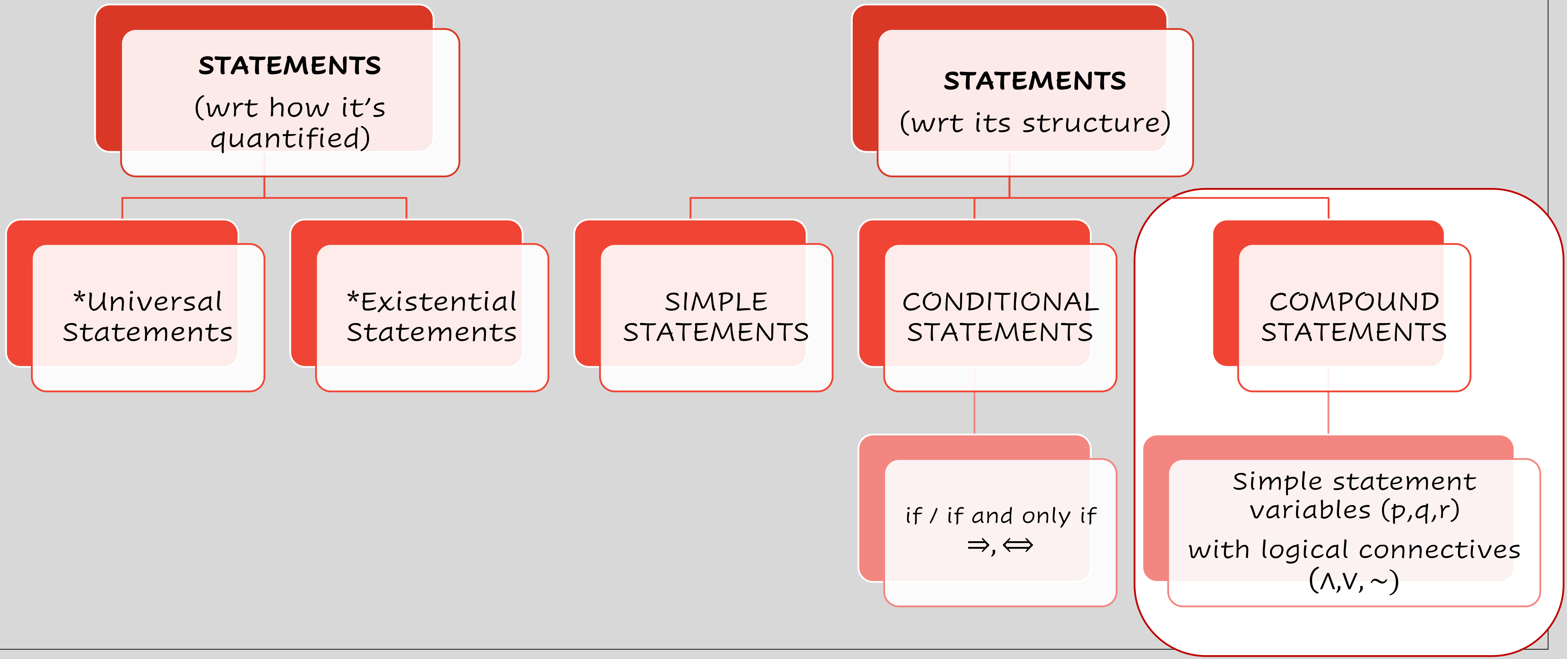
LOGICAL FORMS and LOGICAL EQUIVALENCE

(Logical) Statements (/Propositions):

A logical statement (or proposition) is a sentence that is true OR false, but not both. (T, F)

No questions, no exclamations, enough clarity.

STATEMENTS AND LOGICAL FORMS



Truth Values

- A simple statement has 2 possible truth values.
- ***Negation*** of a statement $\boldsymbol{p}$: $\sim\boldsymbol{p}$

Truth Table →

p	$\sim p$
T	F
F	T

Compound Statements

◦ Includes simple statement variables (p,q,r) connected with logical operations (connectives).

$$\sim, \wedge, \vee, \oplus$$

! Number of possible truth values for a compound statement:

$$2^{\#\ of\ simple\ statement\ variables}$$

! Each simple statement in a compound statement should be *complete*.

- ! Ali is tall and thin. ✕
- ! Ali is tall and Ali is thin. ✓

Compound Statements

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

conjunction

disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Compound Statements

- Truth table for «exclusive or»

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

- Construct a truth table for $(p \wedge q) \vee \sim r$

De Morgan’s Rule

$$\sim(p \wedge q) \equiv \sim p \vee \sim q.$$

$$\sim(p \vee q) \equiv \sim p \wedge \sim q.$$

De Morgan's Rule

Example

- Express the negation of the compound statement $-1 < x \leq 4$, using De Morgan's Rule.

Logical Equivalence

Two statement forms are called **logically equivalent** if, and only if they have identical truth values for each possible substitution of statements for their statement variables(*).

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$$p \wedge q \equiv q \wedge p$$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

$$p \equiv \sim(\sim p)$$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$$\sim (p \wedge q) \not\equiv \sim p \wedge \sim q$$

Tautologies and Contradictions

- A **tautology** is a statement form that is **always true** regardless of the truth values of the individual statements substituted for its statement variables.

a tautological statement -> **t**

- A **contradication** is a statement form that is **always false** regardless of the truth values of the individual statements substituted for its statement variables.

a contradictory statement -> **c**

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F
		↑	↑

Tautologies and Contradictions

p	t	$p \wedge t$	p	c	$p \wedge c$
T	T	T	T	F	F
F	T	F	F	F	F

Some Important Logical Equivalences

$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
$\sim(\sim p) \equiv p$	
$p \wedge p \equiv p$	$p \vee p \equiv p$
$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

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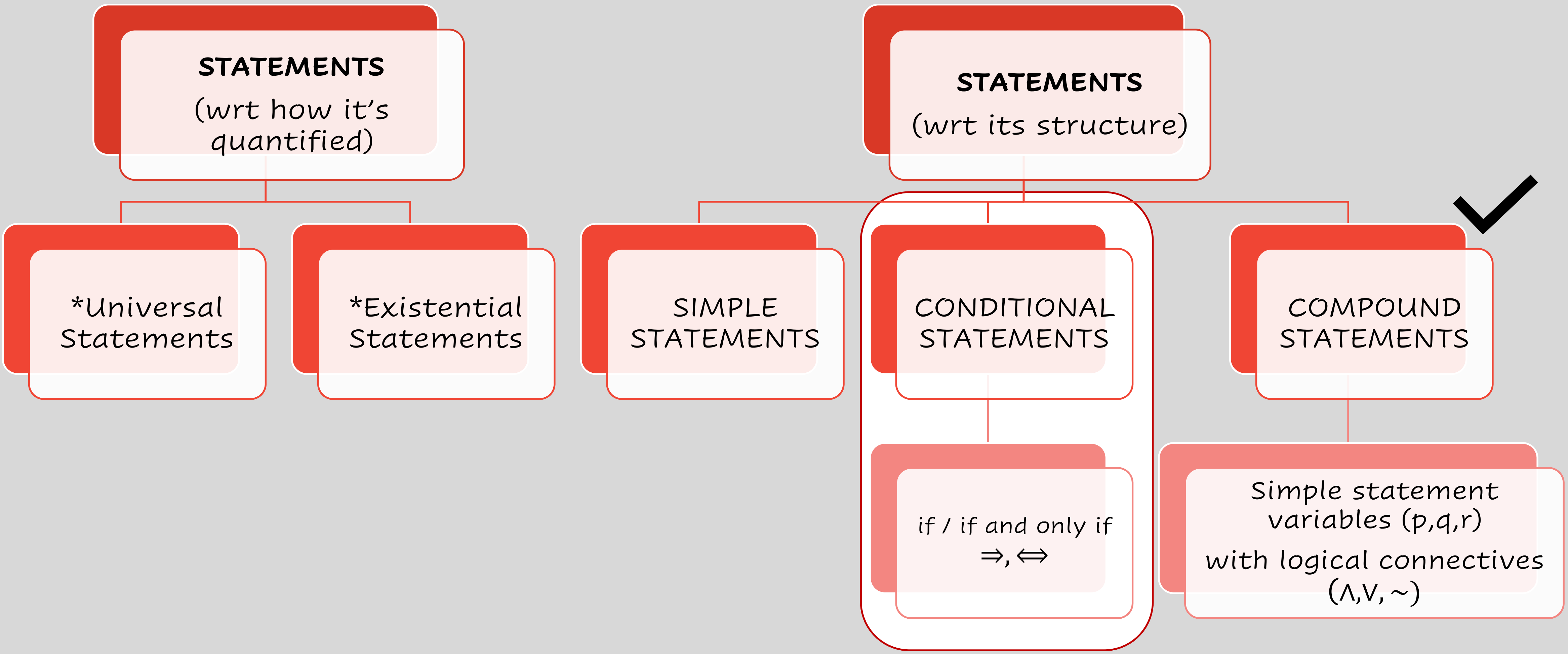


Conditional Statements

- Given p and q as statement variables, the conditional denoted by $p \Rightarrow q$ is read as "**If p then q**" or "**p implies q**".

$$p \Rightarrow q$$

hypothesis   conclusion

A conditional statement that is true by virtue of the fact that its hypothesis is false is often called ***vacuously true*** or ***true by default***.

Conditional Statements

Truth Table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

hypothesis   conclusion

true by default

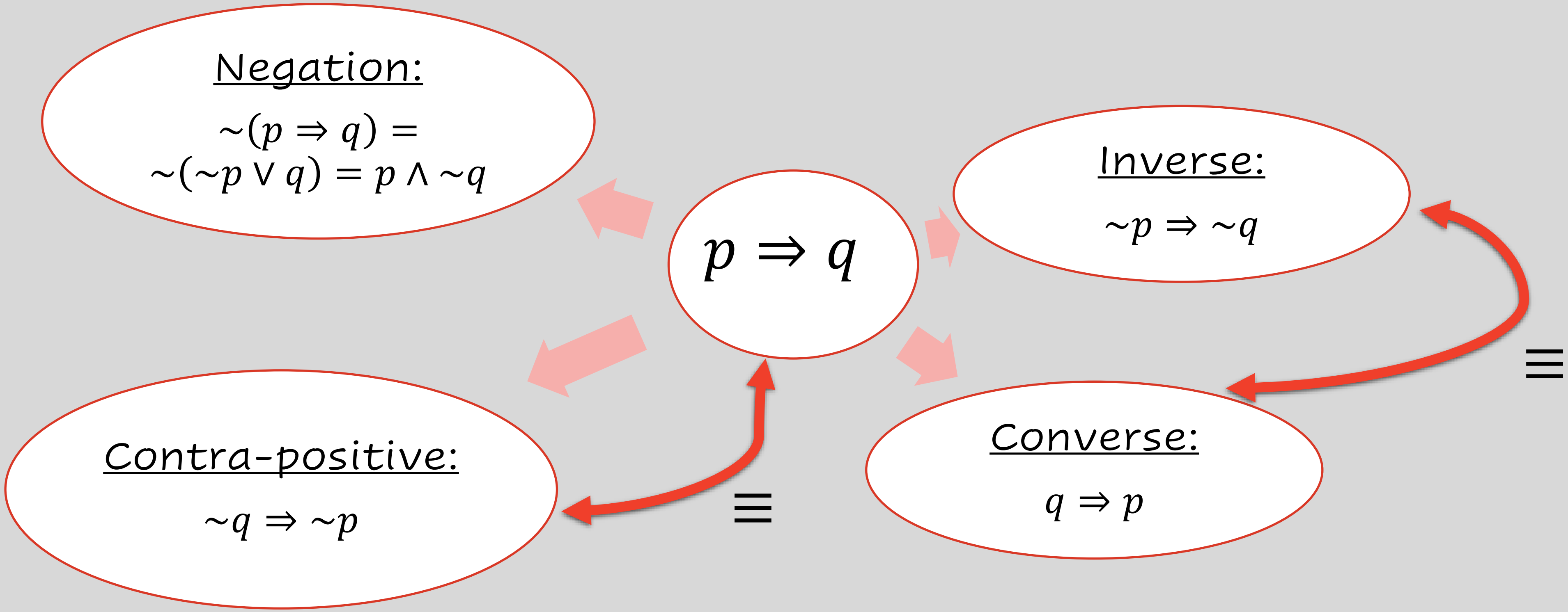
<u>Example:</u>
p: x is a cat.
q: x is cute.
P->q: «If x is a cat then she is cute.»

If x is not a cat (the case where p is false), the statement «If x is a cat then she is cute» is true by default.

<u>Exercise:</u> Construct a truth table for the statement:

$$p \vee \sim q \Rightarrow \sim p$$

Negation, contra-positive, inverse and converse of a conditional statement



! A conditional statement is logically equivalent to its contra-positive.
Converse and inverse of a conditional statement are logically equivalent.

Bi-conditional Statements (if and only if)

$$p \Leftrightarrow q$$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Bi-conditional Statements (if and only if)

r is a **sufficient** condition for s:

«If r, then s»

$r \Rightarrow s$

+

r is a **necessary** condition for s:

«If not r, then not s»

$\sim r \Rightarrow \sim s \equiv s \Rightarrow r$

$$r \Leftrightarrow s$$

Order of Operations for Logical Operators

1. Negations and parantheses
2. $\wedge$, $\vee$
3. $\Rightarrow$, $\Leftrightarrow$

STATEMENTS AND LOGICAL FORMS