* Add / multiply / subtract two integers, the result is an integer.

* If $b \neq 0 \wedge a \neq 0 \Rightarrow ab \neq 0$

$(p \wedge q) \Rightarrow r \equiv$ Contra-positive: $\sim r \Rightarrow \sim(p \wedge q)$

$\sim r \Rightarrow \sim p \vee \sim q$

If $\underset{\sim r}{ab = 0} \Rightarrow$ either $b = 0$ or $a = 0$.

Even/Odd Integers:	Prime Numbers:	Rational Numbers:	Divisibility:
n is even $\Leftrightarrow \exists k \in \mathbb{Z}: n = 2k$ n is odd $\Leftrightarrow \exists k \in \mathbb{Z}: n = 2k+1$	$1 < n$ is prime $\Leftrightarrow (n = rs \Leftrightarrow$ $(r = 1 \wedge s = n) \vee (r = n \wedge s = 1))$	$r \in \mathbb{R}$ is a rational number $\Leftrightarrow$ $(\exists a, b \in \mathbb{Z}: r = a/b \wedge b \neq 0)$	$d \mid n \Leftrightarrow$ $(\exists k \in \mathbb{Z}: n = dk)$

To prove $\underbrace{\forall x \in D, p(x)}_{\text{assume}} \Rightarrow \underbrace{q(x)}_{\text{try to show}}$ → Direct Proof.

negation: $\exists x \in D: p(x) \wedge \sim q(x)$

Disprove the following statement by finding a counterexample: → a specific example for the negation

Ex/ $\forall$ real numbers a and b, if $a^2 = b^2$ then $a = b$.

$\forall a, b \in \mathbb{R}, \; a^2 = b^2 \Rightarrow a = b$. (negation: $\exists a, b \in \mathbb{R}: (a^2 = b^2) \wedge (a \neq b)$)

counter example: $-1, 1 \in \mathbb{R}$, $(-1)^2 = 1^2$ ✓, $-1 \neq 1$ ✓

✓ not enough but necessary ✓ ✓

Ex/ $(\forall a, b \in \mathbb{Z}$, if $3 \mid (a+b) \Rightarrow 3 \mid (a-b))$ neg: $\exists a, b \in \mathbb{Z}: 3 \mid (a+b) \wedge 3 \nmid (a-b)$

does not divide.

$a = 4 \quad b = 2$

counter example: $a = 4, b = 2 \in \mathbb{Z}$, $3 \mid (4+2)$ and $3 \nmid (4-2)$

$6 = 3 \cdot \underset{\in \mathbb{Z}}{2}$ ✓ $\qquad 2 = 3 \cdot k \rightarrow k = \frac{2}{3} \notin \mathbb{Z}$

Ex/ $(\forall a, b \in \mathbb{Z}, a \mid b \wedge b \mid a \Rightarrow a = b)$ → evil person

counter example: $a = -5, b = 5 \in \mathbb{Z}$: $-5 \mid 5$ and $5 \mid (-5)$ and $-5 \neq 5$.

$5 = (-5) \cdot \underset{\in \mathbb{Z}}{(-1)} \qquad -5 = 5 \cdot \underset{\in \mathbb{Z}}{(-1)}$

Proof by Cases

→ a type of direct proof.

$\forall x \in D, (p(x) \vee q(x)) \Rightarrow r(x)$

Proof: assumption case 1: Let $p(x)$ hold.

$\Rightarrow$ ---- $\Rightarrow$ ---- $\Rightarrow r(x)$ ∎

assumption case 2: Let $q(x)$ hold.

$\Rightarrow$ ---- $\Rightarrow$ ---- $\Rightarrow r(x)$. ∎

19. Prove that for all integers n, $n^2 - n + 3$ is odd. ✓

$2k+1$

nothing in common → $n = \text{even}$, $n = \text{odd}$ → the whole space

$\forall n \in \mathbb{Z}, (n = \text{even} \vee n = \text{odd}) \Rightarrow n^2 - n + 3$ is odd.

proof: (proof by cases)

case 1 : Let $n \in \mathbb{Z}$ be even.

$\Rightarrow n = 2k, \; \exists k \in \mathbb{Z}.$

$\Rightarrow n^2 - n + 3 = (2k)^2 - 2k + 3 = 4k^2 - 2k + 3$

$= 2(\underbrace{2k^2 - k + 1}_{\in \mathbb{Z}}) + 1 \Rightarrow n^2 - n + 3$ is odd.

case 2 : Let $n \in \mathbb{Z}$ be odd.

$\Rightarrow n = 2k+1, \; \exists k \in \mathbb{Z}$

$\Rightarrow n^2 - n + 3 = (2k+1)^2 - (2k+1) + 3$

$= 4k^2 + 4k + \cancel{1} - 2k - \cancel{1} + 3 = 4k^2 + 2k + 3$

$= 2(\underbrace{2k^2 + k + 1}_{\in \mathbb{Z}}) + 1 \Rightarrow n^2 - n + 3$ is odd.

Theorem 4.4.1 The Quotient-Remainder Theorem

Given any integer n and positive integer d, there exist unique integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \le r < d.$$

10 | 3
1

n | 3
0, 1, 2 → remainders

$\forall n \in \mathbb{Z}$

n may be written in one of these : $3k, 3k+1, 3k+2$

Ex/

The square of any odd integer has the form $8m + 1$ for some integer m. Proove it.

$n \in \mathbb{Z} \to$ odd

n^2 can be written in the form $8m+1$, $\exists m \in \mathbb{Z}$

required form for n^2.

Let $n \in \mathbb{Z}$ be odd.

$n = 2k+1$

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

→ Divisibility by 4 : $n^2 = 4q$ or $n^2 = 4q+1$ $\quad n^2 = 4q+$

$\Rightarrow$ Divisibility by 4 : ($\cancel{n = 4q}$ $\quad \cancel{n = 4q+2}$ even)

$n = 4q+1$ or $n = 4q+3$

→ case 1 : Let $n = 4q+1$ $\quad \exists q \in \mathbb{Z}$

$\Rightarrow n^2 = (4q+1)^2 = 16q^2 + 8q + 1 = 8(\underbrace{2q^2 + q}_{\in \mathbb{Z}}) + 1$ ✓

→ case 2 : Let $n = 4q+3$, $\exists q \in \mathbb{Z}$

$\Rightarrow n^2 = (4q+3)^2 = 16q^2 + 24q + 9 = 8(\underbrace{2q^2 + 3q + 1}_{\in \mathbb{Z}}) + 1$ ✓