4th Week Friday

27 Ekim 2023 Cuma 09:27

Proof by Contraposition

original → $\forall x \in D: P(x) \Rightarrow Q(x)$

$\equiv$

contra-positive : $\forall x \in D: \sim Q(x) \Rightarrow \sim P(x)$

assume this — try to show this

Proof by Contradiction

original : $\forall x \in D: P(x)$ OR $\forall x \in D: P(x) \Rightarrow Q(x)$

Assume the negation to be true : $\exists x \in D: \sim P(x)$ — $\exists x \in D: P(x) \wedge \sim Q(x)$

$\Rightarrow$ – – (*)

$\Rightarrow$ – (**)

try to find contradictory statements

(*) and (**) contradicts ※

$\therefore$ The originial statement is true.

$\Rightarrow$ – –

$\Rightarrow$ – – –

$\Rightarrow$ – – – –

try to find contradictory statements

→ "

! You may have situations where more than one proof technique may work.

! When do we prefer using contradiction : if the statement talks about lack of sth.

Ex $\forall n \in \mathbb{Z}$, if $3 \nmid n^2 \Rightarrow 3 \nmid n$ — you may choose anyone.

$p \Rightarrow q$

$p \wedge \sim q$

Proof: (by contraposition)

contrapositive : $\forall n \in \mathbb{Z}$, if $3 \mid n \Rightarrow 3 \mid n^2$

assume that $n \in \mathbb{Z}$, $3 \mid n$.

$\Rightarrow n = 3.k$, $\exists k \in \mathbb{Z}$

$\Rightarrow n^2 = (3k)^2 = 9k^2 = 3\underbrace{(3k^2)}_{\in \mathbb{Z}} \Rightarrow 3 \mid n^2$ ■

Proof: (by contradiction)

(neg : $\exists n \in \mathbb{Z}$: $3 \nmid n^2 \wedge 3 \mid n$)

Assume that $n \in \mathbb{Z}$, $3 \nmid n^2$ (*) and $3 \mid n$

$\Rightarrow n = 3k$, $\exists k \in \mathbb{Z}$

$\Rightarrow n^2 = 9k^2 = 3\underbrace{(3k^2)}_{\in \mathbb{Z}} \Rightarrow 3 \mid n^2$ (**)

(*) and (**) contradict. ※

$\Rightarrow$ The original statement is true. ■

Ex $\forall n \in \mathbb{Z}$, $4 \nmid (n^2 - 2)$. Prove that.

proof: (by contradiction)

Assume $\exists n \in \mathbb{Z}$: $4 \mid (n^2 - 2)$.

$\Rightarrow n^2 - 2 = 4k$, $\exists k \in \mathbb{Z}$.

$\Rightarrow n^2 = 4k + 2$ (*)

Remember that by quo-rem. thm (at divisibility by 2)

$\forall n \in \mathbb{Z}$, n can be written in the form

$\Rightarrow n = 2k$ or $n = 2k + 1$

$\Rightarrow$ $n = 2k$ or $n = 2k+1$

for this case; $n^2 = (2k)^2 = 4k^2$

for this case; $n^2 = (2k+1)^2 = 4k^2+4k+1 = 4(\underbrace{k^2+k}_{\in\mathbb{Z}})+1$

$\Rightarrow$ n^2 can be written in the form $4q$ or $4q+1$, $\exists q \in \mathbb{Z}$ (**)

(*) and (**) contradicts.

$\Rightarrow$ The original statement is true ■

r : rational number $\Leftrightarrow$ $r = \frac{a}{b}$, $\exists a,b \in \mathbb{Z}$, $b \neq 0$

if you do all possible cancellations;

$\frac{16}{48} = \frac{4}{12} = \frac{1}{3}$ $\rightarrow r = \frac{a'}{b'}$ $\gcd(a',b') = 1$

Convention: $r = \frac{a}{b}$, $\exists a,b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$

Ex Prove $\sqrt{2}$ is irrational.

Proof. (by contradiction).

Assume $\sqrt{2}$ is rational.

$\Rightarrow$ $\sqrt{2} = \frac{a}{b}$, $\exists a,b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$. (*)

$\sqrt{2} = \frac{16}{48} = \cdots = \frac{1}{3}$

$\Rightarrow$ $a = b\sqrt{2}$ $\Rightarrow$ $a^2 = \underbrace{b^2}_{\in\mathbb{Z}}.2$ $\Rightarrow$ $2 \mid a^2$

by (1) $\Rightarrow$ $2 \mid a$.

(1) we proved before that: $\forall n \in \mathbb{Z}$, if n^2 is even $\Rightarrow$ n is even

$\Rightarrow$ $a = 2k$, $\exists k \in \mathbb{Z}$

$\Rightarrow$ $a^2 = 4k^2 = 2b^2$ $\Rightarrow$ $2k^2 = b^2$ $\Rightarrow$ $2 \mid b^2$

by (1) $\Rightarrow$ $2 \mid b$.

$2 \mid a$ and $2 \mid b$ $\Rightarrow$ 2 is a common divisor of a and b. (**)

(*) and (**) condradict.

Therefore, $\sqrt{2}$ is irrational. ■

try yourself. $\sqrt[3]{2}$ $\sqrt{3}$

Ex Prove that $1+3\sqrt{2}$ is irrational.

proof: (by contradiction)

Assume that $1+3\sqrt{2}$ is rational.

$\Rightarrow 1 + 3\sqrt{2} = \dfrac{a}{b}$, $\exists\, a, b \in \mathbb{Z}$, $b \neq 0$, $\gcd(a,b) = 1$

$\Rightarrow 3\sqrt{2} = \dfrac{a}{b} - 1 \Rightarrow 3\sqrt{2} = \dfrac{a-b}{b}$

$\Rightarrow \sqrt{2} = \dfrac{a-b}{3b}$ $a-b \in \mathbb{Z}$, $3b \in \mathbb{Z}$, $3b \neq 0$ $\Rightarrow \sqrt{2}$ is rational $(*)$

But we already proved that it is irrational $(**)$

$(*)$ and $(**)$ contradict

$\Rightarrow 1 + 3\sqrt{2}$ is irrational. ∎