

Mathematical Induction

$\forall n \geq a, P(n)$

Proof: Basis Step: Show  $P(a)$  to be true.

Inductive Step: Show  $P(k) \Rightarrow P(k+1)$  for some  $k > a$ .

Induction hypothesis: Assume  $P(k)$  is true for some  $k > a$ .

$\Rightarrow$  embed induction hypo  
 $\Rightarrow$  Is  $P(k+1)$  true?

how to prove  $P(k+1)$ ?  
 try to prove

Conclusion:  $\forall n \geq a, P(n)$  is true ■

\* equalities  $\sum_{i=1}^n \dots = \dots = \dots \prod_{i=1}^n \dots = \dots \forall n \geq a$

\* inequalities today

\* divisibility \* sequence

17)  $\prod_{i=1}^n \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}$  for all integers  $n \geq 0$ .

$P(n): \left( \frac{1}{1} \cdot \frac{1}{2} \right) \cdot \left( \frac{1}{3} \cdot \frac{1}{4} \right) \cdot \left( \frac{1}{5} \cdot \frac{1}{6} \right) \dots \left( \frac{1}{2n+1} \cdot \frac{1}{2n+2} \right) = \frac{1}{(2n+2)!} \quad \forall n \geq 0$

Proof: Basis Step:  $n=0$   $\frac{1}{1} \cdot \frac{1}{2} \stackrel{?}{=} \frac{1}{(2 \cdot 0 + 2)!} = \frac{1}{2!}$   $\frac{1}{2} = \frac{1}{2} \checkmark$   
 LHS RHS  $\Rightarrow P(0)$  is true

Inductive Step: Assume that  $P(k)$  holds for some  $k > 0$ .

ind. hypo:  $\left( \frac{1}{1} \cdot \frac{1}{2} \right) \cdot \left( \frac{1}{3} \cdot \frac{1}{4} \right) \dots \left( \frac{1}{2k+1} \cdot \frac{1}{2k+2} \right) = \frac{1}{(2k+2)!}$

for  $n=k+1$ :  $\left( \frac{1}{1} \cdot \frac{1}{2} \right) \left( \frac{1}{3} \cdot \frac{1}{4} \right) \dots \left( \frac{1}{2k+1} \cdot \frac{1}{2k+2} \right) \cdot \left( \frac{1}{2k+3} \cdot \frac{1}{2k+4} \right) = \frac{1}{(2k+2)!} \cdot \left( \frac{1}{2k+3} \cdot \frac{1}{2k+4} \right)$   
 $= \frac{1}{(2k+4)!}$

$= \frac{1}{(2k+2)!} \cdot \frac{1}{(2k+3)} \cdot \frac{1}{(2k+4)} = \frac{1}{(2k+4)!}$

$\Rightarrow P(k+1)$  became true.

Therefore,  $\forall n \geq 0, P(n)$  is true ■

$P(k+1)? \quad n=k+1$   
 $\left( \frac{1}{1} \cdot \frac{1}{2} \right) \dots \left( \frac{1}{2k+1} \cdot \frac{1}{2k+2} \right) \cdot \left( \frac{1}{2(k+1)+1} \cdot \frac{1}{2(k+1)+2} \right) = \frac{1}{(2(k+1)+2)!}$   
 $\downarrow$   
 $\dots \left( \frac{1}{2k+3} \cdot \frac{1}{2k+4} \right) = \frac{1}{(2k+4)!}$   
 $(2k+4)! = (2k+4) \cdot (2k+3) \cdot (2k+2)!$

Showing Inequalities Using Mathematical Induction

**Example**

• Prove for all integers  $n \geq 3$

$$2n+1 < 2^n$$

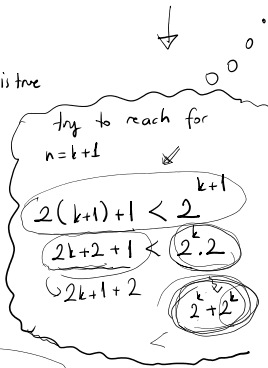
Proof: Basis Step:  $n=3$  is  $P(3)$  true?  $\frac{2 \cdot 3 + 1}{7} < \frac{2^3}{8} \checkmark \Rightarrow P(3)$  is true

Inductive Step: Assume that  $P(k)$  is true for some  $k \geq 3$ . ( $n=k$ )

ind. hyp.  $(2k+1) < 2^k$

Add 2 each side  $\Rightarrow 2k+1 + 2 < 2^k + 2 < 2^k + 2^k = 2^{k+1}$

$\Rightarrow$  since  $k \geq 3 \Rightarrow 2 < 2^k \Rightarrow 2k+3 < 2^{k+1}$



$$\Rightarrow 2(k+1)+1 < 2^{k+1} \Rightarrow P(k+1) \text{ is true.}$$

$\Rightarrow$  conc:  $\forall n \geq 3, 2n+1 < 2^n$

**Example**

• Prove for all integers  $n \geq 5$ ,

$$n^2 < 2^n$$

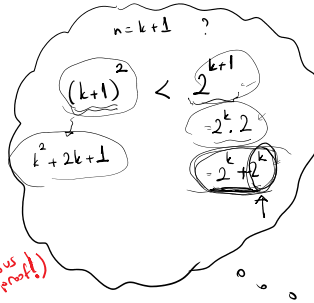
$\Rightarrow$  Basis Step:  $P(5)$ ?  $n=5$   $\frac{5^2}{25} < \frac{2^5}{32} \checkmark \Rightarrow P(5)$  is true.

Inductive Step: Assume  $P(k)$  is true for some  $k \geq 5$ .

ind. hyp.  $k^2 < 2^k$

Add  $2k+1$  both sides  $\Rightarrow k^2 + 2k + 1 < 2^k + 2k + 1 < 2^k + 2^k = 2^{k+1}$

$\Rightarrow (k+1)^2 < 2^{k+1}$  for all  $k \geq 3$  (comes from the previous proof)



$$\Rightarrow (k+1)^2 < 2^{k+1} \Rightarrow P(k+1) \text{ became true.} \blacksquare$$

**Hw**  $\forall a, b, c \in \mathbb{Z}$  if  $a^2 + b^2 = c^2 \Rightarrow$  at least one of  $a$  and  $b$  is even.

(Prove it using some previous proofs as a fact inside)

$\uparrow$   
4<sup>th</sup> week Friday