5th Week Fridays Lecture

3 Kasım 2023 Cuma 09:27

Mathematical Induction

$\forall n \geq a$, $P(n)$

Proof : Basis Step : Show $P(a)$ to be true.

Inductive Step : Show $P(k) \Rightarrow P(k+1)$ for some $k > a$.

induction hypothesis : Assume $P(k)$ is true for some $k>a$.

$\Rightarrow$ embed induction hypo

$\Rightarrow$ Is $P(k+1)$ true?

how to reach $P(k+1)$? try to foresee

Conclusion : $\forall n \geq a$ $P(n)$ is true ■

* equalities $\sum_{i=}^{n} \sim = \sim$ $\prod_{i=}^{n} \sim = \sim$ $\forall n \geq a$
* inequalities today
* divisibility * sequence

17. $\prod_{i=0}^{n} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}$, for all integers $n \geq 0$.

$i=1 \quad i=2$

$i=n$

$P(n)$: $\left(\frac{1}{1}\cdot\frac{1}{2}\right)\cdot\left(\frac{1}{3}\cdot\frac{1}{4}\right)\cdot\left(\frac{1}{5}\cdot\frac{1}{6}\right)\cdots\cdot\left(\frac{1}{2n+1}\cdot\frac{1}{2n+2}\right) = \frac{1}{(2n+2)!}$ $\quad \forall n \geq 0$

Proof : Basis Step: $n=0$ $\quad \underbrace{\frac{1}{1}\cdot\frac{1}{2}}_{LHS} \stackrel{?}{=} \underbrace{\frac{1}{(2.0+2)!}}_{RHS} = \frac{1}{2!}$ $\quad \frac{1}{2} = \frac{1}{2}$ ✓

$\Rightarrow P(0)$ is true

$n=k$

Inductive Step : Assume that $P(k)$ holds for some $k > 0$.

ind. hypo. $\left(\frac{1}{1}\cdot\frac{1}{2}\right)\cdot\left(\frac{1}{3}\cdot\frac{1}{4}\right)\cdots\left(\frac{1}{2k+1}\cdot\frac{1}{2k+2}\right) = \frac{1}{(2k+2)!}$

$P(k+1)$? $n=k+1$

$\left(\frac{1}{1}\cdot\frac{1}{2}\right)\cdots\left(\frac{1}{2k+1}\cdot\frac{1}{2k+2}\right)\cdot\left(\frac{1}{2(k+1)+1}\cdot\frac{1}{2(k+1)+2}\right) = \frac{1}{(2(k+1)+2)!}$

$\cdots\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right) = \frac{1}{(2k+4)!}$

$(2k+4)! = (2k+4).(2k+3).(2k+2)!$

For $n=k+1$:

$\underbrace{\left(\frac{1}{1}\cdot\frac{1}{2}\right)\left(\frac{1}{3}\cdot\frac{1}{4}\right)\cdots\left(\frac{1}{2k+1}\cdot\frac{1}{2k+2}\right)}_{=\frac{1}{(2k+2)!}}\cdot\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right) = \frac{1}{(2k+2)!}\cdot\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right)$

$= \frac{1}{(2k+2)!}\cdot\frac{1}{(2k+3)}\cdot\frac{1}{(2k+4)} = \frac{1}{(2k+4)!}$

$\Rightarrow P(k+1)$ became true.

Therefore , $\forall n \geq 0$ $P(n)$ is true ■

Showing Inequalities Using Mathematical Induction

Example

- Prove for all integers $n \geq 3$

$$2n + 1 < 2^n$$

Proof: Basis Step: $n=3$ is $P(3)$ true? $\underbrace{2.3+1}_{7} < \underbrace{2^3}_{8}$ ✓ $\Rightarrow P(3)$ is true

Inductive Step: Assume that $P(k)$ is true for some $k \geq 3$. $(n=k)$

ind. hypo.: $2k+1 < 2^k$

Add 2 each side; $\Rightarrow 2k+1+2 < 2^k+2 < 2^k + 2^k = 2^{k+1}$

$\Rightarrow$ since $k \geq 3 \Rightarrow 2 < 2^k$

$\Rightarrow 2k+3 < 2^{k+1}$

$\Rightarrow 2(k+1)+1 < 2^{k+1} \Rightarrow P(k+1)$ is true.

try to reach for $n=k+1$

$2(k+1)+1 < 2^{k+1}$

$2k+2+1 < 2^k.2$

$2k+1+2$ $\quad 2^k+2^k$

$\Rightarrow$ Conc.: $\forall n \geq 3,\ 2n+1 < 2^n$

Example

- Prove for all integers $n \geq 5$,

$$n^2 < 2^n$$

$\rightarrow$ Basis Step: $P(5)$? $n=5$ $\quad 5^2 = 25 < 2^5 = 32$ ✓ $\Rightarrow P(5)$ is true.

Inductive Step: Assume $P(k)$ is true for some $k \geq 5$.

ind. hypo.: $k^2 < 2^k$

Add $2k+1$ both sides $\Rightarrow k^2 + 2k+1 < 2^k + 2k+1 < 2^k + 2^k$

$2^k + 2k+1 < 2^k$ $+2^k$ for all $k \geq 3$ (comes from the previous proof!)

$\Rightarrow (k+1)^2 < 2^{k+1} \Rightarrow P(k+1)$ became true. ■

$n = k+1$?

$(k+1)^2 < 2^{k+1}$

$k^2 + 2k+1$ $\quad = 2^k.2$

$= 2^k + 2^k$

HW $\forall a,b,c \in \mathbb{Z}$ if $a^2+b^2=c^2 \Rightarrow$ at least one of a and b is even.

(Prove it using some previous proofs as a fact inside)

4th week Friday