#5.2 Proof by Mathematical Induction

- In this section, Mathematical Induction, as a proof technique, will be covered on some examples of number theoretical concepts and sequences.

Proof by Mathematical Induction

1. Equalities (involving Sums/Product)
2. Inequalities
3. Divisibility
4. Sequences (General Terms, recursive sequences)

Example

- For all integers n≥ 2 , prove that

$$2^n < (n+1)!$$

Example

- Show for all integers $n \geq 0$,

$$3 \mid (2^{2n} - 1)$$

Example

- Show for all integers n ≥ 0,

$$4|(5^n-1)$$

Example

- The recursive definition for a sequence is given by $a_1 = 2, a_k = 5a_{k-1}, \forall k \geq 2.$
- Write out the first four term of the sequence.
- Prove that $a_n = 2.5^{n-1}$, $\forall n \geq 1.$

25. A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all integers $k \geq 1$. Show that $b_n > 4n$ for all integers $n \geq 0$.

Strong Mathematical Induction

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. $P(a), P(a+1), \ldots,$ and $P(b)$ are all true. (**basis step**)
2. For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true. (**inductive step**)

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true. (The supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis.** Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.)

Example

- Any integer greater than 1 is divisible by a prime number.

Example

- The recursive definition for a sequence is given by $a_0 = 0, a_1 = 4, a_k = 6a_{k-1} - 5a_{k-2}, \forall k \geq 2.$
- Write out the first four term of the sequence.
- Prove that $a_n = 5^n - 1\ ,\ \forall n \geq 0.$

8. Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$h_0 = 1, \; h_1 = 2, \; h_2 = 3,$$
$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \quad \text{for all integers } k \geq 3.$$

a. Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.