6th Week Friday

10 Kasım 2023 Cuma 09:31

- → equalities ✓
- → inequalities ✓
- – divisibility ✓
- – sequences

Show

$\forall n \geq a \quad P(n)$

Math. Ind. → Basis Step : $P(a)$?

Ind. Step : Assume $P(k)$ is true for ~~some~~ $k > a$

ind. hypo. → $P(k)$ ✓

→ (Try to show $P(k+1)$ holds.) ←

$a \quad k$

$P(k) \Rightarrow P(k+1)$

Fill in the missing pieces in the following proof that

$1+3+5+\cdots+(2n-1) = n^2$

$\sum_{i=1}^{n}(2i-1) = n^2$

for all integers $n \geq 0$. $n=1$

Proof: Let the property $P(n)$ be the equation

$1+3+5+\cdots+(2n-1) = n^2$

$1 \stackrel{?}{=} 1^2 = 1$

For the basis step: We should show that $P(n)$ is true for: choose $n=1$

To do this, we check the left hand side and right hand side of the equation separately and we see that they are both equal to $\boxed{1}$.

For the inductive step: We assume that $P(k)$ is true for some $k \geq 1$. That is:

$1+3+5+\cdots+\boxed{2k-1} = \boxed{k^2}$ → $n=k$

for $n = k+1$

$\underbrace{1+3+5+\ldots+2k-1}_{=k^2} + 2(k+1)-1 = (k+1)^2$ by the ind. hyp.

For $n=k+1$ we embed the inductive hypothesis inside the left hand side of the equation and we get:

$\boxed{k^2} + (2(k+1)-1) = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k+1)^2$

This equals to $\boxed{}$ $(k+1)^2$, which is exactly what we needed for the right hand side of the equation in this step.

Since we have proved the basis step and the inductive step, we conclude that the given statement is true.

Examples of Math. Ind. Proofs on Sequences

$a_1 = 2$

$a_2 = 5.a_1 = 10$

$a_3 = 5.a_2 = 50$

$a_4 = 5.a_3 = 250$

$\vdots$

Example

- The recursive definition for a sequence is given by $a_1 = 2, a_k = 5a_{k-1}, \forall k \geq 2$. → The question gives you the recursive definition of the sequence ✓
- → Write out the first four term of the sequence.
- Prove that $a_n = 2.5^{n-1}$, $\forall n \geq 1$. → The question asks you to prove the general formula $\forall n \geq a$.

$P(n)$

Proof : Basis Step : Is $P(1)$ true? Can you write $a_1 \stackrel{?}{=} 2.5^{n-1}$ when $n=1$.

$2.5^{1-1} = 2.5^0 = 2$ ✓ it fits to the given rec. defn.

Inductive step : Assume that $P(k)$ is true for some $k \geq 1$.

$\boxed{a_k = 2.5^{k-1}}$

For $n = k+1$; $a_{k+1} = 5.a_k$ (→ write a_{k+1} from the recursive definition

embed

$\Rightarrow a_{k+1} = 5.(2.5^{k-1})$ by the induction hypothesis.

$\Rightarrow a_{k+1} = 2.5^{k-1+1} = 2.5^{k+1-1} \Rightarrow P(k+1)$ is true.

$n = k+1$

$a_{k+1} \stackrel{?}{=} 2.5^{k+1-1}$

Therefore, $a_n = 2.5^{n-1}$ is true for $\forall n \geq 1$

25. A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all integers $k \geq 1$. Show that $b_n > 4n$ for all integers $n \geq 0$.

$\boxed{b_0 = 5}$

$b_k = 4 + b_{k-1}$ for $k \geq 1$. } the recursive definition given in the question

Show $b_n > 4n$, $\forall n \geq 0$ → $P(n)$

$P(k) \Rightarrow P(k+1)$ ✓

Proof : Basis Step : Is $P(0)$ true? $b_0 \stackrel{?}{>} 4.0$

$b_0 = 5$ is already given in the rec. defn.

$5 > 0$ ✓ ⇒ $P(0)$ is true.

Inductive Step: Assume $P(k)$ is true for some $k > 0$.

$\boxed{b_k > 4k}$ → inductive hypothesis

For $n = k+1$: $b_{k+1} = 4 + b_k$ by the recursive definition.

$n = k+1$

$b_{k+1} \overset{?}{>} 4(k+1)$

$4k+4$ ✓

By the ind. hyp. we have → $b_k > 4k$

Add 4 to both sides → $\underbrace{b_k + 4}_{b_{k+1}} > 4k+4 \Rightarrow b_{k+1} > 4k+4 = 4(k+1)$

$\Rightarrow P(k+1)$ is true.

Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a and b be fixed integers with $a \le b$. Suppose the following two statements are true:

1. $P(a), P(a+1), \ldots,$ and $P(b)$ are all true. **(basis step)**
2. For any integer $k \ge b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true. **(inductive step)**

Then the statement

for all integers $n \ge a$, $P(n)$

is true. (The supposition that $P(i)$ is true for all integers i from a through k is called the **inductive hypothesis.** Another way to state the inductive hypothesis is to say that $P(a), P(a+1), \ldots, P(k)$ are all true.)

$P(n)$ $\forall n$

$\underbrace{a \; - - - \; b}_{\text{basis step}}$ k $P(k)$

Assume $\boxed{P(a), P(a+1), \ldots, P(b), \ldots, P(k)}$ are all true. → Induction Hypothesis

Try to show $P(k+1)$ to be true.

Math Ind. $P(k) \Rightarrow P(k+1)$

Strong Math Ind $P(a) \wedge P(a+1) \wedge \ldots \wedge P(k) \Rightarrow P(k+1)$

Example

- The recursive definition for a sequence is given by $a_0 = 0, a_1 = 4, a_k = 6a_{k-1} - 5a_{k-2}, \forall k \ge 2.$
- ~~Write out the first four term of the sequence.~~
- Prove that $a_n = 5^n - 1, \forall n \ge 0.$ → $P(n)$

recursive definition is given

$\boxed{a_0 = 0 \\ a_1 = 4}$ ✓

$a_k = 6a_{k-1} - 5a_{k-2}$ $\forall k \ge 2$.

$a_2 = 6 \cdot a_1 - 5 \cdot a_0 = 24$

$a_3 = 6 \cdot a_2 - 5 \cdot a_1 = 6 \cdot 24 - 5 \cdot 4 = 124$

$P(n)$: $a_n = 5^n - 1$ $\forall n \ge 0$.

Proof (Strong Math. Ind). Basis Step: $a_0 \overset{?}{=} 5^0 - 1$ $1 - 1 = 0$ ✓ $a_1 \overset{?}{=} 5^1 - 1$ $5 - 1 = 4$ ✓

$P(0)$ and $P(1)$ are true.

Inductive Step: → Assume that $P(k)$ is true for all $k > 1$

0 1 k k+1

$P(0)$ $P(1)$

assume all are true show

(Assuming $P(2)$, $P(3)$, $P(4)$, ..., $P(k-2)$, $P(k-1)$, $P(k)$ are all true.)

$a_{k-2} = 5^{k-2} - 1$ ✓ $\quad a_{k-1} = 5^{k-1} - 1$ ✓ $\quad a_k = 5^k - 1$ ✓

$a_{k+1} \overset{?}{=} 5^{k+1} - 1$

$a_{k+1} = 6 \cdot a_k - 5 a_{k-1}$ from the recursive definition.

$a_{k+1} = 6(5^k - 1) - 5(5^{k-1} - 1)$ by inductive hypothesis.

$a_{k+1} = 6 \cdot 5^k - 6 - 5 \cdot 5^{k-1} + 5 = 6 \cdot 5^k - 5^k - 1$

$= 5 \cdot 5^k - 1 = 5^{k+1} - 1$

$\Rightarrow P(k+1)$ is true.

$\therefore P(n)$ is true for all $n \ge 0$