#8 Relations

- ! In this section we will go over **Relations** by means of logical definitions and mathematical proofs.

Relations

<u>Relation:</u> A subset of $A \times B$ is called a «relation». $\mathrm{R} \subseteq A \times B$

If $|A| = \mathrm{m}, |B| = \mathrm{n}$ then,

$|\wp(A \times B)| = 2^{\mathrm{mn}}$

Defining Relations Logically

Example

Let $L \subseteq \mathbb{R} \times \mathbb{R}$.

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R}, \mathrm{x\,L\,y} \Leftrightarrow x < y$$

Example

Let R be a relation on/over $\mathbb{Z}$. ($R \subseteq \mathbb{Z} \times \mathbb{Z}$)

$$\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}, \mathrm{x\,R\,y} \Leftrightarrow x - y \text{ is even.}$$

Some Concepts on Relations

1. Inverse of a relation
2. Representation of relations over $\mathbb{R}$ or $\mathbb{Z}$ on the Cartesian Plane.
3. Directed Graph of a relation
4. Matrix Representation of a Relation

Properties of a Relation

• Definition

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x\ R\ x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, *if* $x\ R\ y$ then $y\ R\ x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, *if* $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

1. R is reflexive	$\Leftrightarrow$	for all x in A, $(x, x) \in R$.
2. R is symmetric	$\Leftrightarrow$	for all x and y in A, *if* $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive	$\Leftrightarrow$	for all x, y and z in A, *if* $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Negations…

Example

- Let R be a relation over $\mathbb{Z}$. ($R \subseteq \mathbb{Z} \times \mathbb{Z}$)

$$\forall x, y \in \mathbb{Z}, \qquad \mathrm{x\,R\,y} \Leftrightarrow 3 | \, x - y$$

Check reflexivity, symmetry, transitivity.

Examples

1. $R_1 = \{(0, 0), (0, 1), (0, 3), (1, 1), (1, 0), (2, 3), (3, 3)\}$
2. $R_2 = \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3)\}$
3. $R_3 = \{(2, 3), (3, 2)\}$
4. $R_4 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$
5. $R_5 = \{(0, 0), (0, 1), (0, 2), (1, 2)\}$
6. $R_6 = \{(0, 1), (0, 2)\}$
7. $R_7 = \{(0, 3), (2, 3)\}$
8. $R_8 = \{(0, 0), (1, 1)\}$

Equivalence Relations

- **Definition**

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Equivalence Classes:

- **Definition**

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a**, denoted $[a]$ and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x \; R \; a\}$$

Lemma

Let A be a set and R an equivalence relation on A. If $a\,R\,b$ then $[\mathrm{a}] = [b]$.

Lemma

Let A be a set and R an equivalence relation on A, if $\mathrm{a}, \mathrm{b} \in A$ then,

$[\mathrm{a}] \cap [b] = \emptyset$ or $[\mathrm{a}] = [b]$.

$$p \Rightarrow (q \vee r) \equiv (p \wedge \sim q) \Rightarrow r$$

Theorem

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A.

(Partition):

$[A_1, A_2, \ldots, A_n]$, is a partition for A $\Leftrightarrow$

$(A_i \cap A_j = \emptyset, \forall i \neq j\ ) \wedge (\bigcup_{i=1}^{n} A_i = A\ )$

Examples

In each of 3–14, the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

3. $A = \{0, 1, 2, 3, 4\}$
   $R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$

4. $A = \{a, b, c, d\}$
   $R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$