

Relations $R \subseteq A \times B$

R, M, L, T



ex Let M be a relation on \mathbb{Z} defined by $(x, y) \in M$
 $M \subseteq \mathbb{Z} \times \mathbb{Z} \quad \forall x, y \in \mathbb{Z}, \quad x M y \Leftrightarrow 3 | (x - y)$

$x M y \Rightarrow y M x$
 $x M y \Rightarrow y M x \rightarrow$

Properties: Reflexive: $\forall x \in A, x R x \quad (x, x) \in R$ negations \exists counter example

$p \Rightarrow q$ Symmetry: $x R y \Rightarrow y R x \quad \forall x, y \in A$ \exists counter exam $x R y \quad y \not R x$

$(p \wedge q) \Rightarrow r$ Transitivity: $x R y \wedge y R z \Rightarrow x R z \quad \forall x, y, z \in A$

Equivalence Relations: \checkmark Ref \wedge \checkmark sym \wedge \checkmark transitivity

Equivalence Classes: $[a] = \{x \in A : x R a\}$
 $R \subseteq A \times A \quad a \in A$

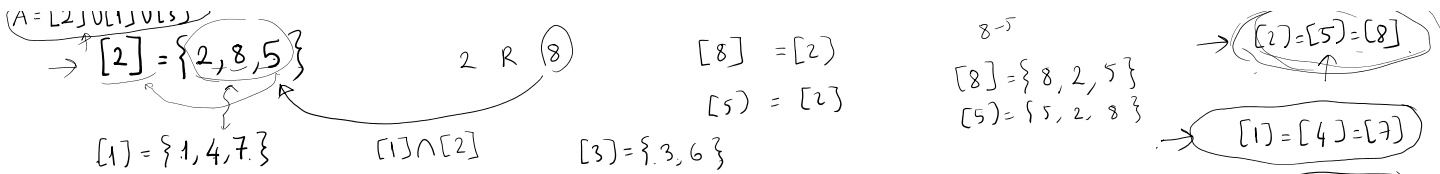
Lemma

Let A be a set and R an equivalence relation on A . If $a R b$ then $[a] = [b]$.

Proof: Let $a R b$. $[a] = \{x, \dots\}$ $[b] = \{ \dots \}$
 part 1: $[a] \subseteq [b]$ \rightarrow prove it
 Let $x \in [a] \Rightarrow x R a \quad x R a \quad a R b$
 together with $a R b$ $\Rightarrow x R b \Rightarrow x \in [b]$
 and by transitivity of R
 part 2: $[b] \subseteq [a]$
 Let $x \in [b] \Rightarrow x R b$
 $a R b$ since R is sym. $b R a$
 since R is transitive $x R a \Rightarrow x \in [a]$

Therefore $[a] = [b]$

ex $A = \{1, 2, 3, 4, 5, 6, 7, 8\} \quad R \subseteq A \times A \quad \forall x, y \in A \quad x R y \Leftrightarrow 3 | (x - y)$
 $A = [2] \cup [1] \cup [3]$
 $\rightarrow [2] = \{2, 8, 5\} \quad 2 R 8$
 $[8] = [2] \quad [5] = [2]$
 $[8] = \{8, 2, 5\}$
 $\rightarrow [2] = [5] = [8]$



Lemma

Let A be a set and R an equivalence relation on A , if $a, b \in A$ then,

$[a] \cap [b] = \emptyset$ or $[a] = [b]$ \equiv $(a, b \in A \wedge [a] \cap [b] \neq \emptyset) \Rightarrow [a] = [b]$

$p \Rightarrow (q \vee r) \equiv (p \wedge \sim q) \Rightarrow r$

proof: Let $a, b \in A$ and let $[a] \cap [b] \neq \emptyset$.

$\rightarrow x \in [a] \cap [b] \Rightarrow x \in [a] \wedge x \in [b]$
 $\Rightarrow x R a \wedge x R b$

R is symm. $\Rightarrow a R x \wedge x R b$

R is transitivity $\Rightarrow a R b \Rightarrow [a] = [b]$.
From the previous lemma

Theorem

If A is a set and R is an equivalence relation on A , then the distinct equivalence classes of R form a partition of A .

$A = [a_1] \cup [a_2] \cup \dots \cup [a_n]$

(Partition):

$[A_1, A_2, \dots, A_n]$ is a partition for $A \Leftrightarrow$
 $(A_i \cap A_j = \emptyset, \forall i \neq j) \wedge (\cup_{i=1}^n A_i = A)$

distinct equivalence class
 $[a_i] \neq [a_j]$

from the previous lemma
 $[a_i] \cap [a_j] = \emptyset$

To prove $A = [a_1] \cup [a_2] \cup \dots \cup [a_n]$

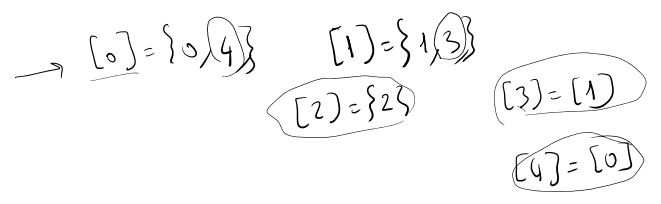
$n \rightarrow$ number of distinct equivalence classes in R

(\subseteq) : Let $a \in A$ $a R a \Rightarrow a \in [a] \rightarrow$ one of the distinct equivalence classes
 $\Rightarrow a \in [a_1] \cup [a_2] \cup \dots \cup [a_n]$

(\supseteq) : Let $a \in [a_1] \cup \dots \cup [a_n]$
 Since they are disjoint $\Rightarrow a \in [a_i]$ only one of them.
 $[a_i] \subseteq A \Rightarrow a \in A$

In each of 3-14, the relation R is an equivalence relation on the set A . Find the distinct equivalence classes of R .

- 3. $A = \{0, 2, 3, 4\} = [0] \cup [1] \cup [2]$
 $R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$



- 4. $A = \{a, b, c, d\}$
 $R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$