9th Week Friday

1 Aralık 2023 Cuma 09:25

Relations $R \subseteq A \times B$

(x,y), $x \in A$, $y \in B$

R, M, L, T

ex/ Let M be a relation on $\mathbb{Z}$ defined by

$M \subseteq \mathbb{Z} \times \mathbb{Z}$ $\forall x,y \in \mathbb{Z}$, $x\,M\,y \Leftrightarrow 3 \mid (x-y)$ $(x,y) \in M$

$x\,M\,y$ $\quad$ $x\,M\,y \Rightarrow y\,M\,x \rightarrow$

$y\,M\,x$

Properties: Reflexive : $\forall x \in A$ $\quad x\,R\,x$ $\quad (x,x) \in R$ $\quad \exists$ negations counter example

$R \subseteq A \times A$

$p \Rightarrow q$ Symmetry : $x\,R\,y \Rightarrow y\,R\,x$ $\quad \forall x,y \in A$ $\quad \exists$ counter exam $x\,R\,y$ $y \not R\, x$

$(p \wedge q) \Rightarrow r$ Transitivity : $x\,R\,y \wedge y\,R\,z \Rightarrow x\,R\,z$ $\quad \forall x,y,z \in A$

Equivalence Relations : Ref $\wedge$ sym $\wedge$ transitivity

Equivalence Classes : $[a] = \{x \in A : x\,R\,a\}$

$R \in A \times A$ $\quad a \in A$

Lemma

Let A be a set and R an equivalence relation on A. If $a\,R\,b$ then $[a]=[b]$.

ref, sym, transit.

$[b] = \{ \quad \}$

Proof: Let $a\,R\,b$. $\quad [a] = \{x, \quad \}$

part 1 : $[a] \subseteq [b]$ → prove it $\quad x \in [b]$ $\quad x\,R\,b$

Let $x \in [a]$. $\Rightarrow x\,R\,a$ $\quad x\,R\,a$ $\quad a\,R\,b$

together with $a\,R\,b$ $\Rightarrow x\,R\,b. \Rightarrow x \in [b]$.

and by transitivity of R

Therefore $[a] = [b]$

part 2 : $[b] \subseteq [a]$ $\quad x \in [a]$ $\quad a\,R\,x$ $\quad x\,R\,a$

Let $x \in [b]$. $\Rightarrow x\,R\,b$

$a\,R\,b$ since R is sym. $\quad b\,R\,a$ $\xrightarrow{\text{since R is transitive}}$ $x\,R\,a$

$\Rightarrow x \in [a]$.

$x\,R\,b$ $\quad b\,R\,a$

ex/ $A = \{1,2,3,4,5,6,7,8\}$ $\quad R \subseteq A \times A$ $\quad \forall x,y \in A$ $\quad x\,R\,y \Leftrightarrow 3 \mid (x-y)$ equivalence relation (ref, sym, trans)

$2-2=0$

$A = [2] \cup [1] \cup [3]$

→ $[2] = \{2,8,5\}$ $\quad 2\,R\,8$ $\quad [8] = [2]$ $\quad 8-5$ $\quad [8] = \{8,2,5\}$ → $[2]=[5]=[8]$

$[5] = [2]$

$A = [2] \cup [1] \cup [3]$

$\rightarrow [2] = \{2, 8, 5\}$ $\quad 2 \; R \; 8$ $\quad [8] = [2]$ $\quad [5] = [2]$

8 - 5

$[8] = \{8, 2, 5\}$

$[5] = \{5, 2, 8\}$

$\rightarrow [2] = [5] = [8]$

$[1] = \{1, 4, 7\}$ $\quad [1] \cap [2]$ $\quad [3] = \{3, 6\}$

$[1] = [4] = [7]$

$[3] = [6]$

$p \Rightarrow (q \vee r) \equiv (p \wedge \sim q) \Rightarrow r$

Lemma

Let A be a set and R an equivalence relation on A, if $p \rightarrow$ $a, b \in A$ then, $\underbrace{[a] \cap [b] = \emptyset}_{q}$ or $\underbrace{[a] = [b]}_{r}$. $\equiv$ $(a, b \in A \wedge [a] \cap [b] \neq \emptyset) \Rightarrow [a] = [b]$

proof: Let $a, b \in A$ and let $[a] \cap [b] \neq \emptyset$.

$\rightarrow x \in [a] \cap [b] \Rightarrow x \in [a] \wedge x \in [b]$

$\Rightarrow x R a \wedge x R b$

R is symm. $\Rightarrow a R x \wedge x R b$

R is transitivity $\Rightarrow a R b \Rightarrow [a] = [b]$.

From the previous lemma

Theorem

$A = \{ \quad \}$

$\Rightarrow$ If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A. $\quad A = [a_1] \cup [a_2] \cup \ldots [a_n]$

(Partition):

$[A_1, A_2, \ldots, A_n]$, is a partition for A $\Leftrightarrow$ $(A_i \cap A_j = \emptyset, \forall i \neq j) \wedge (\cup_{i=1}^{n} A_i = A)$

distinct equivalence class

$[a_i] \neq [a_j]$ $\Rightarrow$ $[a_i] \cap [a_j] \neq \emptyset$

from the previous lemma

To prove $A = [a_1] \cup [a_2] \cup \ldots \cup [a_n]$ $\quad$ $n \rightarrow$ number of distinct equivalence classes in R

$(\subseteq)$: let $a \in A$ $\quad \underset{\text{reflexive}}{a R a} \Rightarrow a \in [a] \rightarrow$ one of the distinct equivalence classe $\Rightarrow a \in [a_1] \cup [a_2] \cup \ldots \cup [a_n]$

$(\supseteq)$: Let $a \in [a_1] \cup \ldots \cup [a_n]$

Since they are disjoint $\Rightarrow a \in [a_i]$ only one of them. $\quad [a_i] \subseteq A \Rightarrow a \in A$

In each of 3–14, the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

3. $A = \{0, 1, 2, 3, 4\} = [0] \cup [1] \cup [2]$
   $R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$

$\rightarrow [0] = \{0, 4\}$ $\quad [1] = \{1, 3\}$ $\quad [2] = \{2\}$ $\quad [3] = [1]$ $\quad [4] = [0]$

4. $A = \{a, b, c, d\}$
   $R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$